

Computer Graphics

# Chapter 05

# 3D Transformations

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# Outline

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- 3D Translation
- 3D Rotation
- 3D Scaling
- Other Transformation

# 3D Transformation

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- Now we come to transformations in 3D, where we apply the same reasoning as in 2D.
- Scaling and translation are basically the same, but where in 2D we rotated a shape about a point, in 3D rotation, we rotate an object about an axis.

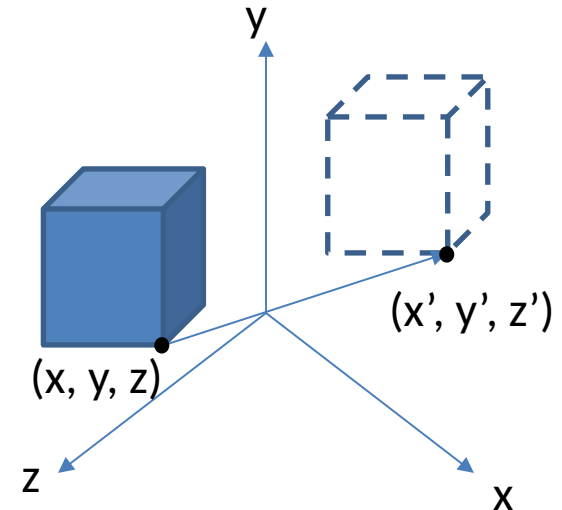
# 3D Translation

- Similar to 2D translation, which used  $3 \times 3$  matrices, 3D translation use  $4 \times 4$  matrices  $(x, y, z, h)$ .
- In 3D translation point  $(x, y, z)$  is to be translated by amount  $t_x$ ,  $t_y$  and  $t_z$  to location  $(x', y', z')$ .

$$x' = x + tx, \quad y' = y + ty, \quad z' = z + tz$$

- Matrix equation,

$$P' = T \cdot P = \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



# Example- 3D Translation

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- Translate the given point  $P(10,10,10)$  into 3D space with translation factor  $T(10,20,5)$ .

$$P' = T \cdot P$$

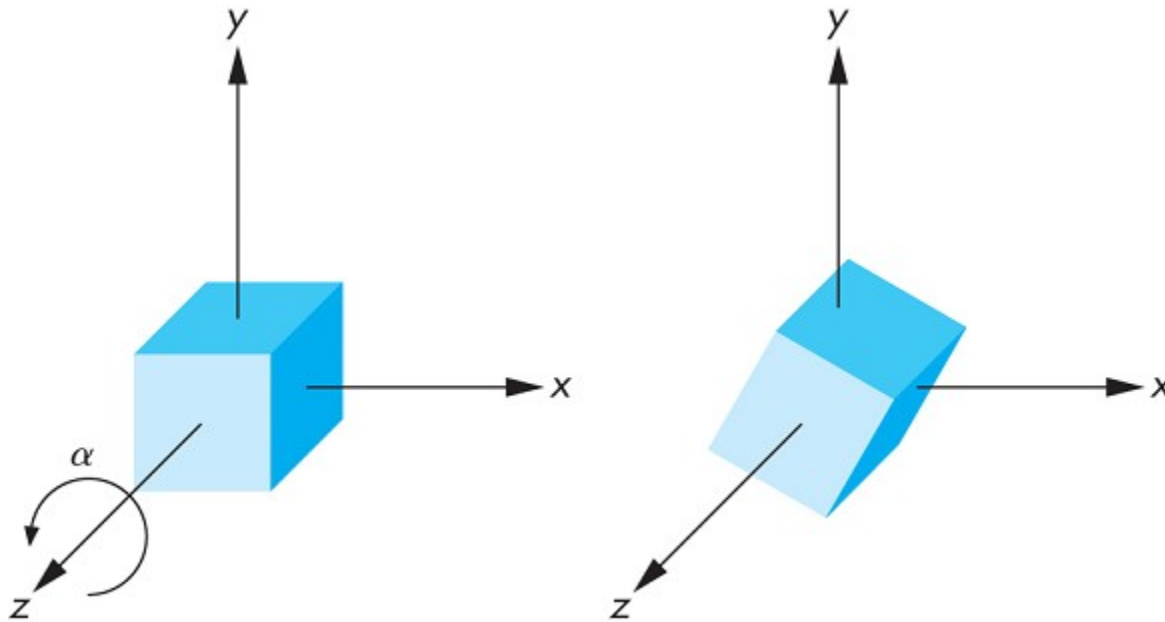
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & 20 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 10 \\ 10 \\ 10 \\ 1 \end{bmatrix} = \begin{bmatrix} 20 \\ 30 \\ 15 \\ 1 \end{bmatrix}$$

- Final coordinate after translation is  $P' (20,30,15)$ .

# Rotation

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- For 3D rotation we need to pick an axis to rotate about.
- The most common choices are the  $X$  – *axis*, the  $Y$  – *axis*, and the  $Z$  – *axis*, it is known as coordinate axis rotation.
- We can also chose other arbitrary axis for rotation.



Source: <http://www.c-jump.com>

# Z-Axis Rotation

- Two dimension rotation equations can be easily convert into 3D *Z – axis* rotation equations.
- Rotation about *z* axis we leave *z* coordinate unchanged.

$$x' = x \cos \theta - y \sin \theta$$

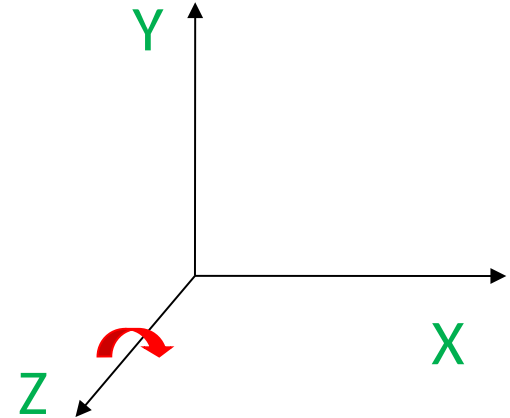
$$y' = x \sin \theta + y \cos \theta$$

$$z' = z$$

where Parameter  $\theta$  specify rotation angle.

- Matrix equation is written as,

$$P' = R_z(\theta) \cdot P = \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta & 0 & 0 \\ \sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



# X-Axis Rotation

- Transformation equation for  $X$  – axis is obtain from equation of  $Z$  – axis rotation by replacing cyclically as  $x \rightarrow y \rightarrow z \rightarrow x$
- Rotation about  $X$  – axis we leave  $x$  coordinate unchanged.

$$y' = y \cos \theta - z \sin \theta$$

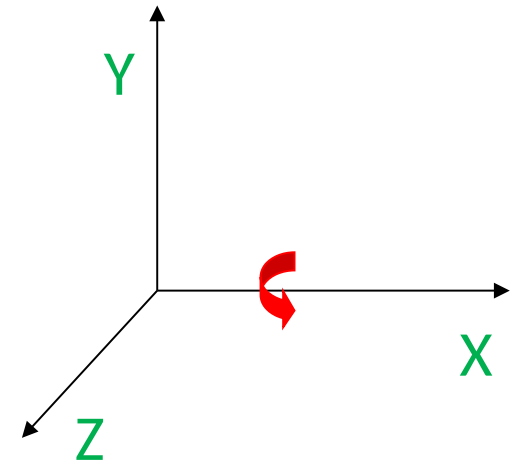
$$z' = y \sin \theta + z \cos \theta$$

$$x' = x$$

where Parameter  $\theta$  specify rotation angle.

- Matrix equation is written as,

$$P' = R_x(\theta) \cdot P = \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta & 0 \\ 0 & \sin \theta & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



# Y-Axis Rotation

- Transformation equation for  $Y$  –  $axis$  is obtain from equation of  $X$  –  $axis$  rotation by replacing cyclically as  $x \rightarrow y \rightarrow z \rightarrow x$
- Rotation about  $Y$  –  $axis$  we leave  $y$  coordinate unchanged.

$$z' = z \cos \theta - x \sin \theta$$

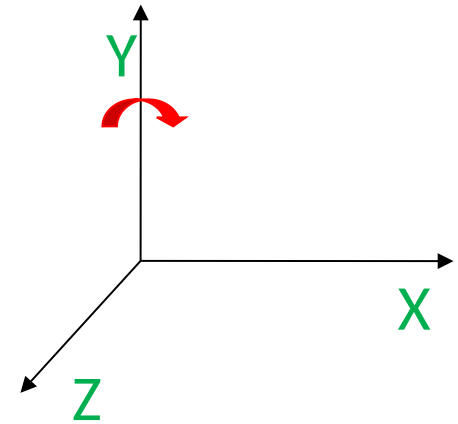
$$x' = z \sin \theta + x \cos \theta$$

$$y' = y$$

where Parameter  $\theta$  specify rotation angle.

- Matrix equation is written as,

$$P' = R_y(\theta) \cdot P = \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos \theta & 0 & \sin \theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin \theta & 0 & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



# Example- Coordinate Axis Rotation

- Rotate the point  $P(5, 5, 5)$   $90^\circ$  about Z-axis.

$$P' = R_z(\theta = 90^\circ) \cdot P$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos 90^\circ & -\sin 90^\circ & 0 & 0 \\ \sin 90^\circ & \cos 90^\circ & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 5 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 5 \\ 5 \\ 1 \end{bmatrix}$$

- Final coordinate after rotation is  $P'(-5, 5, 5)$ .

# General 3D Rotations

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When rotation axis is parallel to one of the standard axis.

- Three steps require to complete such rotation these are,
  1. Translate the object so that the rotation axis coincides with the parallel coordinate axis.
  2. Perform the specified rotation about that axis.
  3. Translate the object so that the rotation axis is moved back to its original position.
- This can be represented in equation form as,

$$P' = T^{-1} \cdot R(\theta) \cdot T \cdot P$$

# General 3D Rotations

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## When rotation axis is inclined in arbitrary direction.

- First we need rotations to align the axis with a selected coordinate axis and to bring the axis back to its original orientation.
- Five steps require to complete such rotation these are,
  1. Translate the object so that the rotation axis passes through the coordinate origin.
  2. Rotate the object so that the axis of rotation coincides with one of the coordinate axes.
  3. Perform the specified rotation about that coordinate axis.
  4. Apply inverse rotations to bring the rotation axis back to its original orientation.
  5. Apply the inverse translation to bring the rotation axis back to its original position.

# Contd.

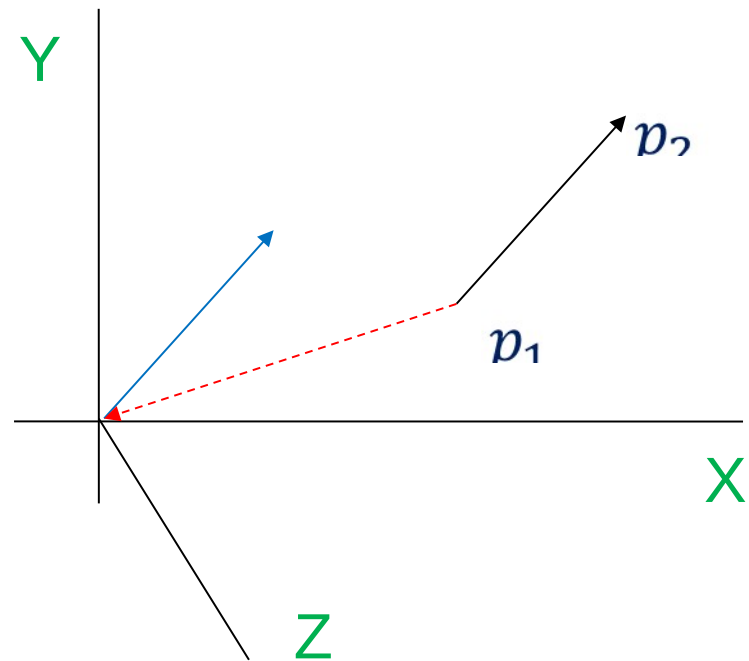
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- We can transform rotation axis onto any of the three coordinate axes. The  $Z$  – *axis* is a reasonable choice.
- We are given line in the form of two end points  $p_1 (x_1, y_1, z_1)$ , and  $p_2 (x_2, y_2, z_2)$ .
- Let's discuss procedure step by step.

# 1. Translate the Object so that the Rotation Axis Passes Through the Coordinate Origin

- For translation of step one we will bring first end point at origin and transformation matrix for the same is as below

$$T = \begin{bmatrix} 1 & 0 & 0 & -x_1 \\ 0 & 1 & 0 & -y_1 \\ 0 & 0 & 1 & -z_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



## 2. Rotate the Object so that the Axis of Rotation Coincides with One of the Coordinate Axes

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- This task can be completed by two rotations first rotation about  $X - axis$  and second rotation about  $Y - axis$ .
- But here we do not know rotation angle so we will use dot product and vector product.
- Vector notation for rotation axis is,  
$$V = P_2 - P_1 = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$
- Unit vector along rotation axis is obtained by dividing vector by its magnitude.

$$u = \frac{V}{|V|} = \left( \frac{x_2 - x_1}{|V|}, \frac{y_2 - y_1}{|V|}, \frac{z_2 - z_1}{|V|} \right) = (a, b, c)$$

# Contd.

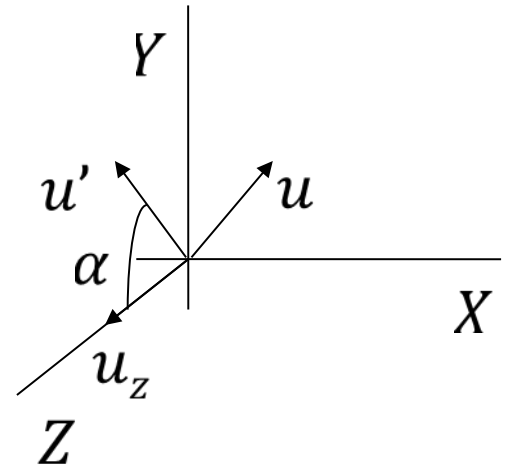
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- Now we need cosine and sine value of angle between unit vector  $u$  and  $XZ$  plane.
- For that take projection of  $u$  on  $YZ$  – plane say  $u'$ .
- Find dot product and cross product of  $u'$  and  $u_z$ .
- Coordinate of  $u'$  is  $(0, b, c)$  as we will take projection on  $YZ$  – plane  $x$  value is zero.
- Dot product,

$$u' \cdot u_z = |u'| |u_z| \cos \alpha$$

$$\cos \alpha = \frac{u' \cdot u_z}{|u'| |u_z|} = \frac{(0, b, c)(0, 0, 1)}{\sqrt{b^2 + c^2}} = \frac{c}{d}$$

$$\text{where } d = \sqrt{b^2 + c^2}$$



# Contd.

- Cross product,

$$u' \times u_z = u_x |u'| |u_z| \sin \alpha \dots (1)$$

$$u' \times u_z = u_x \cdot b \dots (2)$$

- From (1) and (2),

$$u_x |u'| |u_z| \sin \alpha = u_x \cdot b$$

- Comparing magnitude

$$|u'| |u_z| \sin \alpha = b$$

$$\sqrt{b^2 + c^2} \cdot (1) \sin \alpha = b$$

$$d \sin \alpha = b$$

$$\sin \alpha = \frac{b}{d}$$

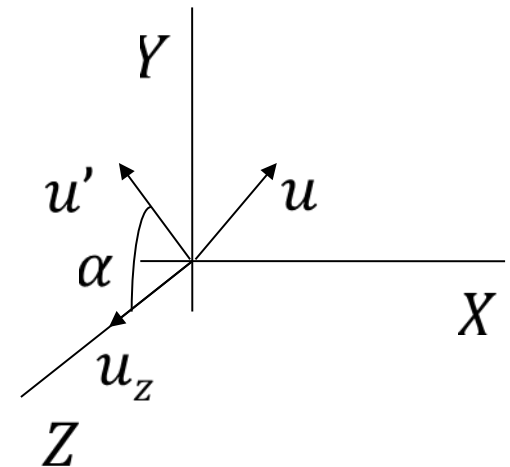
$$u' (0, b, c)$$

$$u_z (0, 0, 1)$$

$$|u' \times u_z| = (b \cdot 1 - c \cdot 0, c \cdot 0 - 0 \cdot 1, 0 \cdot 0 - 0 \cdot b)$$

$$= (b, 0, 0)$$

$$|u' \times u_z| = b$$



# Contd.

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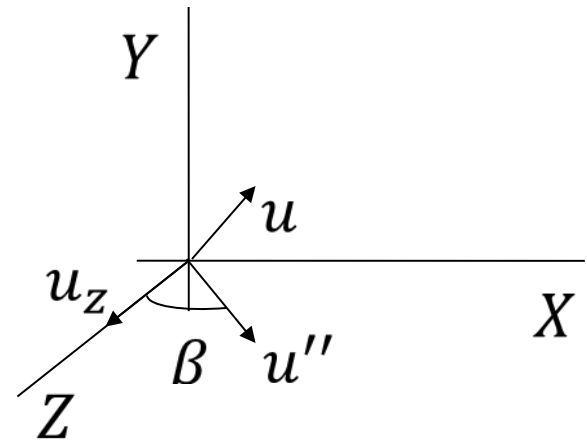
- Now we have  $\sin \alpha$  and  $\cos \alpha$  so we will write matrix for rotation about X-axis.

$$R_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha & 0 \\ 0 & \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{c}{d} & -\frac{b}{d} & 0 \\ 0 & \frac{b}{d} & \frac{c}{d} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

# Contd.

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- After performing above rotation  $u$  will be rotated into  $u''$  in  $XZ$  – *plane* with coordinates  $(a, 0, \sqrt{(b^2 + c^2)})$ .
- As we know rotation about  $x$  axis will leave  $x$  coordinate unchanged.
- $u''$  is in  $XZ$  – *plane* so  $y$  coordinate is zero, and  $z$  component is same as magnitude of  $u'$ .
- Now rotate  $u''$  about  $Y$  – *axis* so that it coincides with  $Z$  – *axis*.



# Contd.

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- For that we repeat above procedure between  $u''$  and  $u_z$  to find matrix for rotation about  $Y - axis$ .

- Dot product,

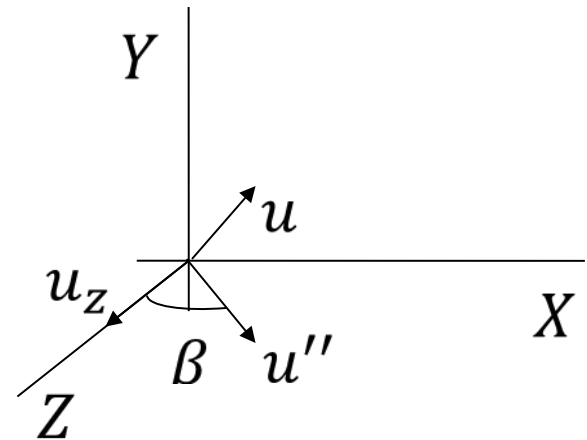
$$u'' \cdot u_z = |u''| |u_z| \cos \beta$$

$$\cos \beta = \frac{u'' \cdot u_z}{|u''| |u_z|}$$

$$\cos \beta = \frac{(a, 0, \sqrt{b^2 + c^2})(0, 0, 1)}{1}$$

$$\cos \beta = \sqrt{b^2 + c^2} = d$$

$$\text{where } d = \sqrt{b^2 + c^2}$$



# Contd.

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- Cross product,

$$u'' \times u_z = u_y |u''| |u_z| \sin \beta \dots (1)$$

$$u'' \times u_z = u_y \cdot (-a) \dots (2)$$

- From (1) and (2),

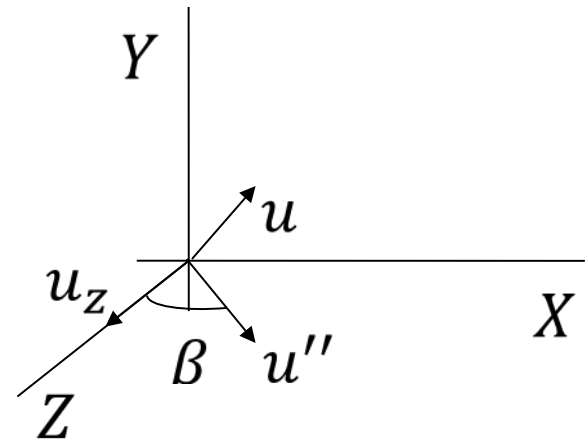
$$u_y |u''| |u_z| \sin \beta = u_y \cdot (-a)$$

- Comparing magnitude

$$|u''| |u_z| \sin \beta = (-a)$$

$$(1) \sin \beta = -a$$

$$\sin \beta = -a$$



# Contd.

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- Now we have  $\sin \beta$  and  $\cos \beta$  so we will write matrix for rotation about  $Y - axis$ .

$$R_y(\beta) = \begin{bmatrix} \cos \beta & 0 & \sin \beta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin \beta & 0 & \cos \beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} d & 0 & -a & 0 \\ 0 & 1 & 0 & 0 \\ a & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Now by combining both rotation we can coincides rotation axis with Z-axis

$$R_y(\beta) \cdot R_x(\alpha) = \begin{bmatrix} d & 0 & -a & 0 \\ 0 & 1 & 0 & 0 \\ a & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{c}{d} & -\frac{b}{d} & 0 \\ 0 & \frac{b}{d} & \frac{c}{d} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

# 3. Perform the Specified Rotation About that Coordinate Axis

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- As we align rotation axis with  $Z$  – *axis* so now matrix for rotation about  $Z$  – *axis*,

$$R_z(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta & 0 & 0 \\ \sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where  $\theta$  is specified rotation angle

## 4. Apply Inverse Rotations to Bring the Rotation Axis Back to it's Original Orientation

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- This step is inverse of step number 2,

$$R_x^{-1}(\alpha) \cdot R_y^{-1}(\beta)$$

# 5. Apply the Inverse Translation to Bring the Rotation Axis Back to it's Original Position

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- This step is inverse of step number 1,

$$T^{-1} = \begin{bmatrix} 1 & 0 & 0 & x_1 \\ 0 & 1 & 0 & y_1 \\ 0 & 0 & 1 & z_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

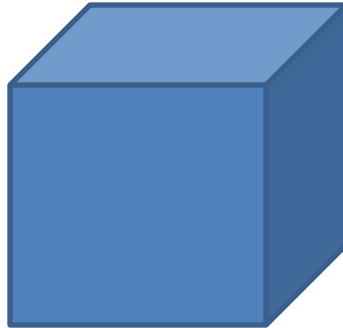
- **So finally sequence of transformation for general 3D rotation is**

$$P' = T^{-1} \cdot R_x^{-1}(\alpha) \cdot R_y^{-1}(\beta) \cdot R_z(\theta) \cdot R_y(\beta) \cdot R_x(\alpha) \cdot T \cdot P$$

# Scaling

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- It is used to resize the object in 3D space.
- We can apply uniform as well as non uniform scaling by selecting proper scaling factor.
- Scaling in 3D is similar to scaling in 2D. Only one extra coordinate need to consider into it.

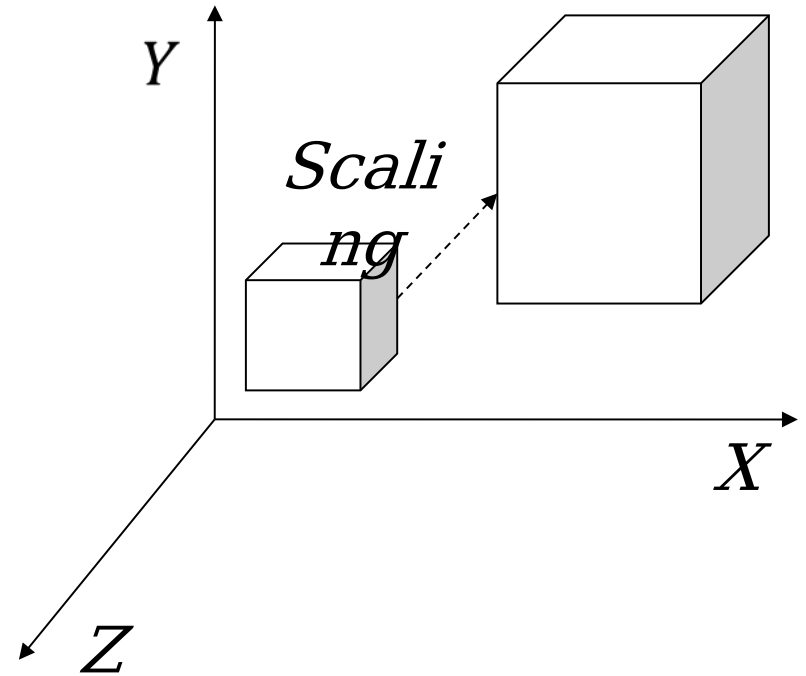


# Coordinate Axes Scaling

- Simple coordinate axis scaling can be performed as below,

$$P' = S \cdot P$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



# Example-Coordinate Axes

## Scaling

- Example: - Scale the line  $AB$  with coordinates  $(10,20,10)$  and  $(20,30,30)$  respectively with scale factor  $S(3,2,4)$ .

$$P' = S \cdot P = \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

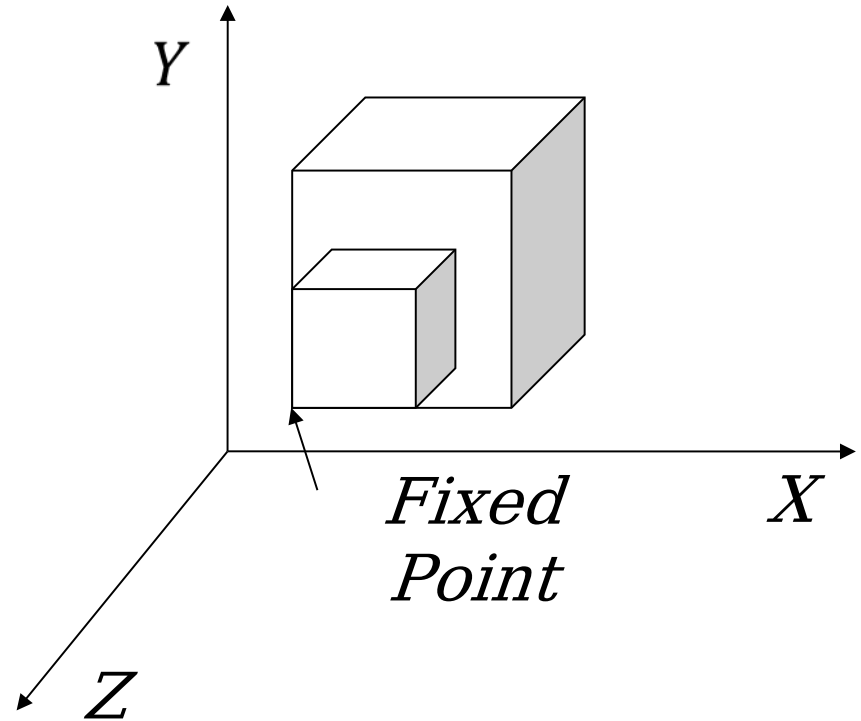
$$\begin{bmatrix} A_x' & B_x' \\ A_y' & B_y' \\ A_z' & B_z' \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 10 & 20 \\ 20 & 30 \\ 10 & 30 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 30 & 60 \\ 40 & 60 \\ 40 & 120 \\ 1 & 1 \end{bmatrix}$$

- Final coordinates after scaling are,

$$A' (30, 40, 40) \text{ and } B' (60, 60, 120)$$

# Fixed Point Scaling

- Fixed point scaling is used when we require scaling of object but particular point must be at its original position.
- Three steps require to complete such fixed point scaling these are,
  1. Translate the fixed point to the origin.
  2. Scale the object relative to the coordinate origin using coordinate axes scaling.
  3. Translate the fixed point back to its original position.



# Contd.

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- Matrix equation

$$P' = T(x_f, y_f, z_f) \cdot S(s_x, s_y, s_z) \cdot T(-x_f, -y_f, -z_f) \cdot P$$

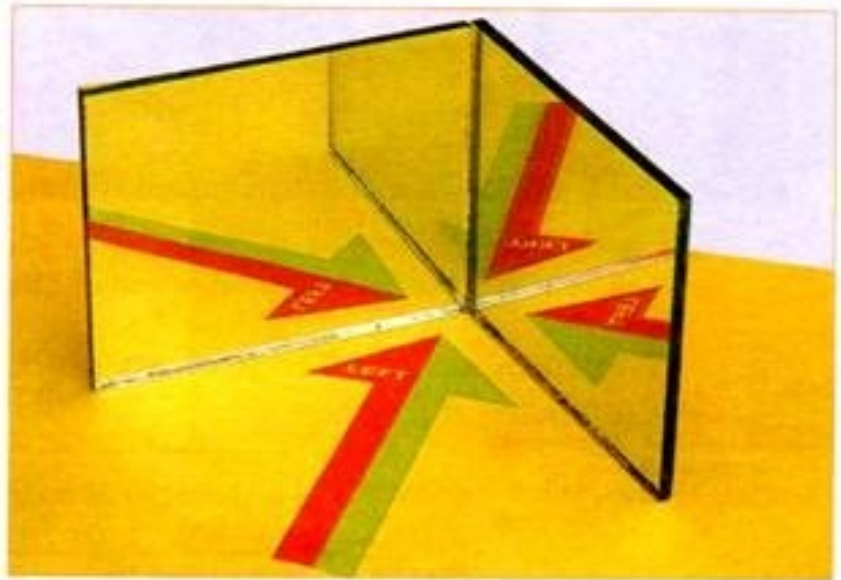
$$P' = \begin{bmatrix} 1 & 0 & 0 & x_f \\ 0 & 1 & 0 & y_f \\ 0 & 0 & 1 & z_f \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & -x_f \\ 0 & 1 & 0 & -y_f \\ 0 & 0 & 1 & -z_f \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot P$$

$$P' = \begin{bmatrix} s_x & 0 & 0 & (1 - s_x)x_f \\ 0 & s_y & 0 & (1 - s_y)y_f \\ 0 & 0 & s_z & (1 - s_z)z_f \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot P$$

# Other Transformations- Reflection

- Reflection means mirror image produced when mirror is placed at require position.
- When mirror is placed in XY-plane we obtain coordinates of image by just changing the sign of z coordinate.
- Transformation matrix for reflection about XY-plane is given below,

$$RF_z = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Source: <http://www.yourarticlelibrary.com>

# Contd.

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- Similarly Transformation matrix for reflection about YZ-plane is,

$$RF_x = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

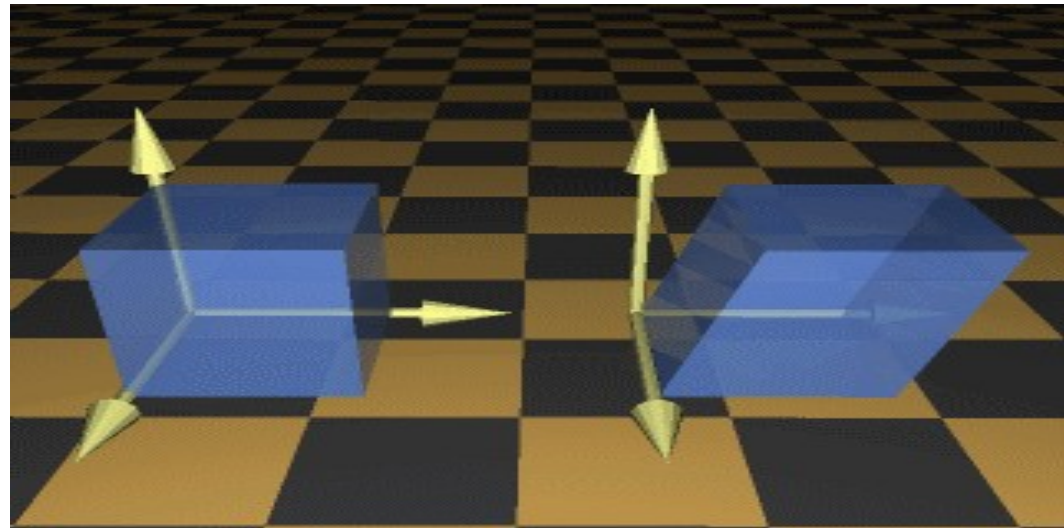
- Similarly Transformation matrix for reflection about XZ-plane is,

$$RF_y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

# Other Transformations-Shear

- Shearing transformation can be used to modify object shapes.
- They are also useful in 3D viewing for obtaining general projection transformations.
- Here we use shear parameter ' $a$ ' and ' $b$ '
- Shear matrix for Z-axis is given below,

$$SH_z = \begin{bmatrix} 1 & 0 & a & 0 \\ 0 & 1 & b & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Source: <http://140.129.20.249/~jmchen/cg>

# Other Transformations-Shear

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- Similarly Shear matrix for X-axis is,

$$SH_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ b & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Similarly Shear matrix for Y-axis is,

$$SH_y = \begin{bmatrix} 1 & a & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & b & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

**Thank You**